

CLASSIFYING FOLIATIONS OF 3-MANIFOLDS VIA
BRANCHED SURFACES

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ABSTRACT. We use branched surfaces to define an equivalence relation on C^1 codimension one foliations of any closed orientable 3-manifold that are transverse to some fixed nonsingular flow. There is a discrete metric on the set of equivalence classes with the property that foliations that are sufficiently close (up to equivalence) share important topological properties.

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Introduction

Dating back to their introduction by Williams in 1969 [Wi1], branched manifolds have been powerful tools in the study of the dynamics of foliations. The one dimensional case, branched 1-manifolds or *train tracks*, were introduced earlier to study Anosov diffeomorphisms [Wi2] and were used by Thurston to describe the dynamics of surface automorphisms [Th]. Branched surfaces were constructed by Williams to study the dynamics of hyperbolic expanding attractors for C^1 diffeomorphisms of compact 3-manifolds [Wi3] and have since been used to obtain many important results in the theory of foliations and laminations of 3-manifolds (e.g. [Ag-Li], [Br], [Ga], [Ga-Ka], [Ga-Oe], [Oe]).

Here we use branched surfaces to define an equivalence relation on C^1 codimension one foliations of any closed orientable 3-manifold that are transverse to a fixed nonsingular flow ϕ . Specifically, we use branched surfaces to define a pseudometric on this set of foliations and then let two foliations be *b-equivalent* if the distance between them, in this pseudometric, is zero. In this way, we obtain a metric with the property that foliations that are sufficiently close (up to b-equivalence) often share important topological properties, such as the existence of a compact leaf, tautness or the property of being R-covered.

To define the pseudometric on foliations transverse to ϕ we introduce a notion of *N-equivalence* for every natural number N. Intuitively, the largest number N for which two foliations F and G are N-equivalent (if such an upper bound exists) indicates the extent of similarity in the ways F and G, with air blown into their leaves, embed in a regular neighborhood of the same branched surface. The greater this number is, the closer F and G will be in our pseudometric.

In Section I, we review Christy and Goodman's [Ch-Go] construction of a branched surface W from a foliation F, a nonsingular transverse flow ϕ , and a *generating set* consisting of compact surfaces embedded in leaves of F that satisfy certain general position requirements with respect to ϕ . In Section II, we discuss techniques for modifying the

branched surface W by extending or contracting some of the elements in its generating set. Each of these modifications either splits W along some smoothly immersed surface with boundary or is the inverse of such an operation (i.e. pinches two pieces of W together along such two such surfaces). In Section III, we use our modification techniques to define b-equivalence for foliations transverse to ϕ . The structural implications of b-equivalence (and N-equivalence, for N sufficiently large) are discussed in Section IV .

I. Branched surfaces constructed from foliations

Throughout this paper, F will be a C^1 codimension one foliation of a closed orientable Riemannian 3-manifold M , and $\phi: M \times \mathbb{R} \rightarrow M$ will be a C^1 nonsingular flow on M that is transverse to F . We shall often refer to the *forward (backward) orbit* of a point $x = \phi(x, 0)$ in M under ϕ . By this, we shall mean the set of points $\phi(x, t)_{t > 0}$ ($\phi(x, t)_{t < 0}$ respectively).

Branched surface construction. The branched surfaces we construct from the foliation F are in the class of regular branched surfaces introduced by Williams. Since the construction we use is in an unpublished paper of Christy and Goodman [Ch-Go] and is a variation of the one in [Ga-Oe], we describe it here, including all details necessary for this article.

We begin by choosing a *generating set for (F, ϕ)* ; that is, we choose a finite set $\Delta = \{D_i\}_{i=1, \dots, n}$ of disjoint embedded compact surfaces with boundary satisfying the following general position requirements:

(i) each D_i is contained in a leaf of F (hence is transverse to ϕ) and has finitely many boundary components,

ii) the forward and backward orbit of every point, under ϕ , meets $\text{int} \Delta = \bigcup_{i=1}^n \text{int} D_i$

iii) the set of points in $\partial\Delta = \bigcup_{i=1}^n \partial D_i$ whose forward orbit meets $\partial\Delta$ before meeting

$\text{int}\Delta$ is finite, and

iv) the forward orbit of any point in $\partial\Delta$ meets $\partial\Delta$ at most once before meeting $\text{int}\Delta$.

Note that it is always possible to choose a generating set satisfying these conditions. In particular, cover M with finitely many foliation boxes for F that are also flow boxes for ϕ , and select a slice from each box. Then, modify each slice so that the resulting collection of disks satisfies the general position requirements above. In cases, such as this, where the generating set consists of embedded disks, we say that it is *standard*.

After choosing a generating set Δ for (F, ϕ) , we cut M open along $\text{int}\Delta$ to obtain a closed submanifold M^* which is embedded in M so that its boundary contains $\partial\Delta$. This can be thought of as blowing air into the leaves of F to create an air pocket at each element of the generating set. By requirement (ii) above, the restriction of ϕ to M^* is a flow ϕ^* with the property that each orbit is homeomorphic to the unit interval $[0, 1]$. We can form a quotient space by identifying points that lie on the same orbit of ϕ^* . That is, we take the quotient M^*/\sim , where $x \sim y$ if x and y lie on the same interval orbit of ϕ^* . The quotient W is a *branched surface carrying F and ϕ* (or *carrying (F, ϕ)*). If Δ is standard, then we say *the branched surface W is standard*.

Note that the branched surface W could have many generating sets. For example, if we flow a disk in Δ forward or backward slightly to another sufficiently close disk, the quotient space described above does not change.

The branched surface W arises as an abstract quotient space. However, it can be realized as a compact connected 2-dimensional complex embedded in M with a set of charts defining local orientation preserving diffeomorphisms onto one of the models in Figures 1.1, 1.2, 1.3 and 1.4 such that the transition maps are smooth and preserve the transverse orientation indicated by the arrows. (In general, the particular embedding used will not be of concern. However, we note that we can choose this embedding so that it is transverse to

ϕ . Specifically, we can view the quotient map as enlarging the components of $M-M^*$ until each interval orbit of ϕ^* is contracted to a point in M .)

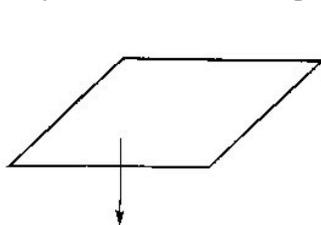


Figure 1.1

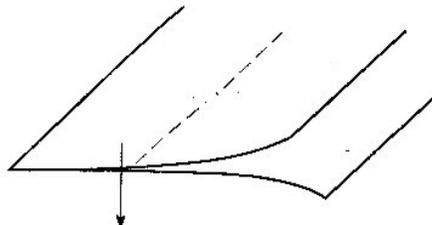


Figure 1.2

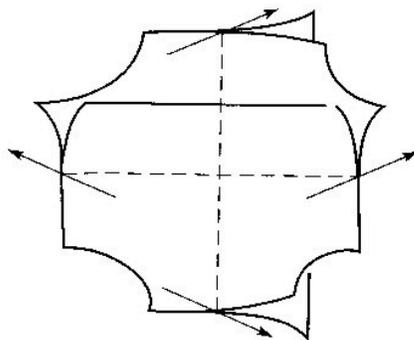


Figure 1.3

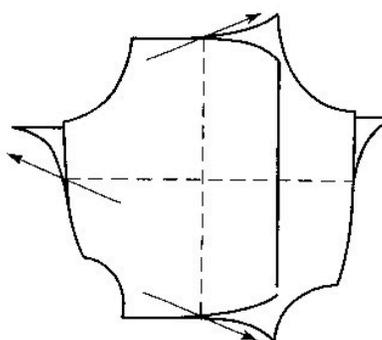


Figure 1.4

Each local model above projects horizontally onto a vertical model of \mathbb{R}^2 . Hence, W has a smooth structure induced by TR^2 when we pull back each local projection. Furthermore, it is a connected 2-manifold except on a dimension one subset μ called the *branch set*. The set μ is a 1-manifold except at finitely many isolated points called *triple points* where it intersects itself transversely. Each component of $W-\mu$ is called a *sector* of W . Note that if W is generated by planar surfaces (i.e. surfaces without handles) then each of its sectors is a planar surface.

We can thicken the branched surface W in the transverse direction to get a manifold $N(W)$ that is diffeomorphic to M^* . More precisely, $N(W)$ is obtained by replacing each point $x \in W$ with a copy I_x of the unit interval $[0,1]$ so that there exists a diffeomorphism from $N(W)$ onto M^* mapping I_x onto the orbit of ϕ^* whose quotient is x . For each $x \in W$, we call I_x the *fiber of $N(W)$ over x* . (See Figure 1.5.) Throughout, $\pi_w: N(W) \rightarrow W$ will

denote the quotient map that identifies points in same fiber. We say the image x of a point under this map is the *projection* of that point.

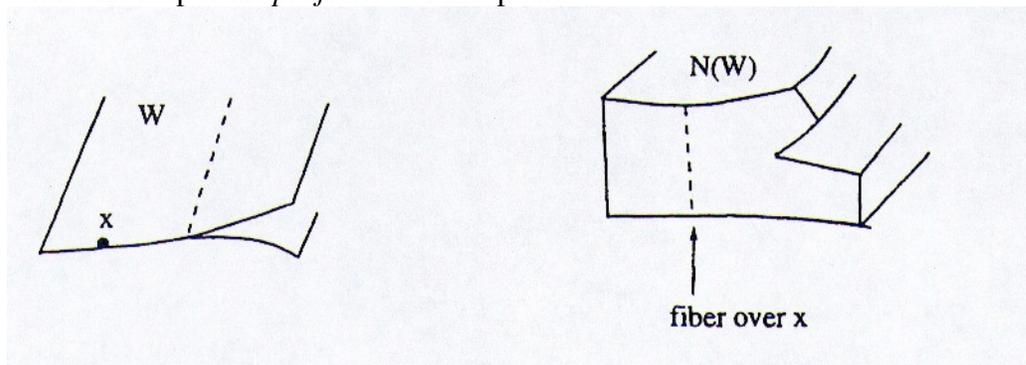


Figure 1.5

Since W could have more than one generating set, we shall henceforth use $N_{\Delta}(W)$ rather than M^* to denote the embedded copy of $N(W)$ that is obtained by cutting M open along a generating set Δ ; in particular, $\partial N_{\Delta}(W)$ contains $\partial\Delta$.

Foliations carried by a branched surface. The foliation F clearly induces a foliation of $N_{\Delta}(W)$ with leaves transverse to the fibers. In particular, the leaves containing the boundary components of $N_{\Delta}(W)$ are precisely the (cut-open) leaves of the original foliation containing the elements of Δ . They can be thought of as leaves of the original foliation with air blown into them. Figure 1.6 shows a local picture of the corresponding foliation of $N(W)$. Each boundary component of $N(W)$ is contained in a branched leaf of this foliation.

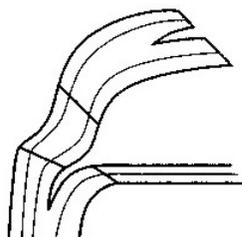


Figure 1.6

It is worth noting that changing generating sets for W can change the foliation of $N(W)$ induced by F . In fact, we can often find two such foliations for which there is no diffeomorphism of $N(W)$ that maps one onto the other and preserves the fibers. So for clarity, we shall use F_Δ to denote the foliation of $N_\Delta(W)$ (as well as the corresponding foliation of $N(W)$) induced by F when we cut M open along $\text{int}\Delta$.

Now, there are many foliations that are transverse to the fibers of $N(W)$ with the property that every boundary component of $N(W)$ is contained in a leaf. Each corresponds to a foliation of $N_\Delta(W)$ that becomes a foliation of M carried by W when we collapse the components of $M-N_\Delta(W)$ (i.e., the air pockets).

We say the branched surface W has a topological property P if every foliation carried by W has property P .

Curves in W . Formally, a curve in M is a continuous map from a connected subset of \mathbb{R} into M . However, we shall consider a *curve* to be the image of such a map, where the map parameterizes the curve. We say a curve is *finite* if this map can be chosen so that its domain is the unit interval $[0,1]$. The *beginning and end* of a finite curve $\gamma(t)_{0 \leq t \leq 1}$ refer to $\gamma(0)$ and $\gamma(1)$ respectively. The *length* of a finite curve $\gamma(t)_{0 \leq t \leq 1}$ contained in the branch set μ of W shall be the cardinality of the set $\{t: 0 \leq t \leq 1 \text{ and } \gamma(t) \text{ is a triple point of } W\}$. Throughout, we shall only consider finite curves in μ whose lengths are minimal in their respective (fixed point) μ -homotopy classes.

Given two foliations F and G carried by W , we say F *shadows* G in W if there exist foliations of $N(W)$ induced by F and G respectively such that for every finite integral curve $\gamma(t)_{0 \leq t \leq 1}$ of F that begins at a point $x \in \partial N(W)$, there exists an integral curve $\gamma'(t)_{0 \leq t \leq 1}$ of G that also begins at x and has the property that $\gamma(t)$ and $\gamma'(t)$ are contained in the same fiber of $N(W)$ for every $0 \leq t \leq 1$.

II Modifications of W

In this section, we describe several techniques for modifying a branched surface carrying (F, ϕ) by changing its generating set. We use these techniques in Section III to define our equivalence relation on the set of foliations transverse to a ϕ .

Given a branched surface W carrying (F, ϕ) with generating set Δ , we can modify W by extending an element D of Δ to include some compact integral surface E of F such that $\partial E \cap \partial D \neq \emptyset$, $\text{int} E \cap \text{int} D = \emptyset$, $\partial(E \cup D) \neq \emptyset$ and $\partial(E \cup D)$ has finitely many components. This, in turn, enlarges the component B of $M-N(W)$ corresponding to D . We refer to this type of modification of D as an *F-extension*. If the new Δ is, in fact, another generating set for (F, ϕ) , then the *F-extension* corresponds to an *F-splitting* of W along the projection $\pi_W(E)$ of E . See Figure 2.1.

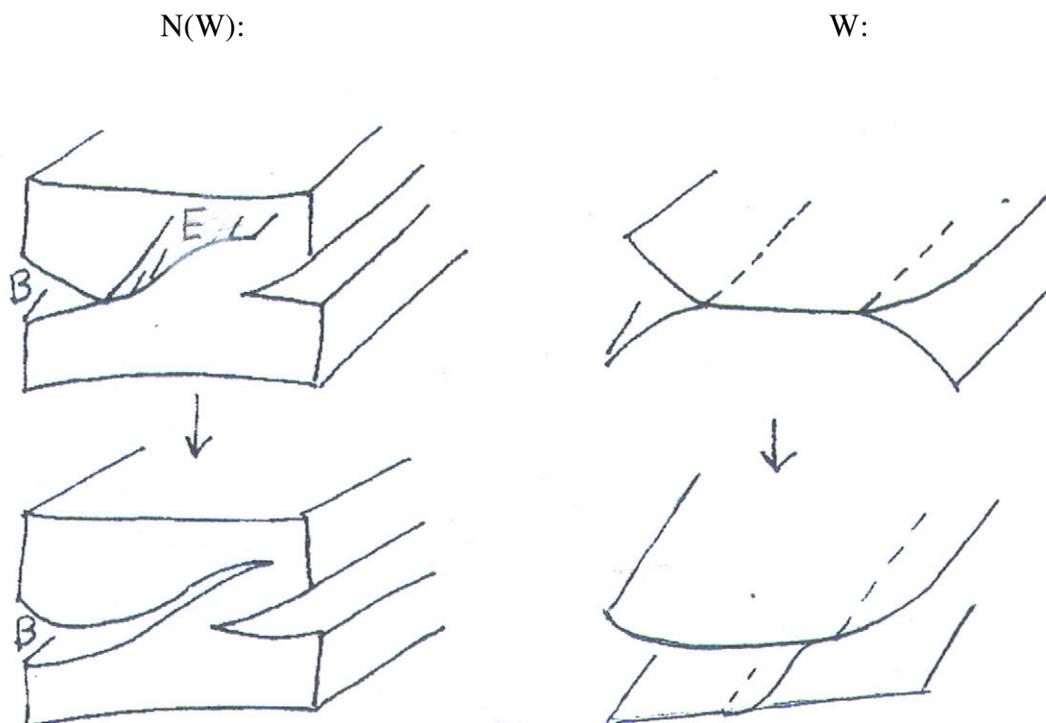


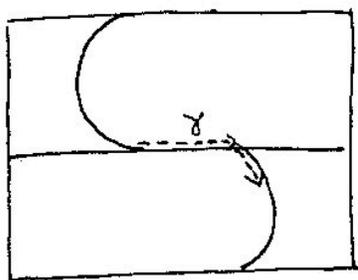
Figure 2.1

Since changing generating sets for W could change the corresponding foliation of $N(W)$ induced by F , *F-extensions* of different generating sets for W can result in different

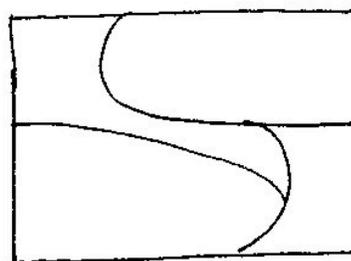
F-splittings of W . So when we refer to an F-splitting of W , we shall mean that for some generating set Δ of W , an F-extension of Δ causes that splitting of W .

Clearly, we can extend $D \in \Delta$ to include any compact surface E embedded transverse to the fibers of $N_\Delta(W)$ whose boundary meets ∂D . This too corresponds to a splitting of W along the projection $\pi_W(E)$ of E . However, if E is not an integral surface of F , then there is no guarantee that this splitting yields a branched surface carrying F . We illustrate with a lower dimensional example.

The branched 1-manifold W in Figure 2.2 carries a foliation F of T^2 with 2 Reeb components and 2 compact leaves. Yet, when we modify W by splitting along the curve γ (indicated by the dashed line), we obtain the branched 1-manifold W' which does not carry F .



W embedded in a planar model of T^2



W' obtained by splitting W along γ

Figure 2.2

So it is worth noting that when we split a branched surface W carrying F , the branched surface we obtain carries F precisely when this splitting is an F-splitting.

We shall mostly consider F-extensions of a generating surface $D \in \Delta$ to include a closed collar neighborhood E of some integral curve of F in its leaf. In particular, we focus on the case where the corresponding integral curve $\gamma_{F_\Delta}(t)_{0 \leq t < 1}$ of F_Δ is contained in the interior of $N_\Delta(W)$, its initial point $\gamma_{F_\Delta}(0)$ is contained in $\partial N_\Delta(W)$ (specifically, $\gamma_{F_\Delta}(0) \in \partial D$), and its projection $\gamma(t)_{0 \leq t \leq 1} = \pi_W(\gamma_{F_\Delta}(t)_{0 \leq t \leq 1})$ onto W is contained in the branch set μ . (Each curve $\gamma(t)_{0 \leq t \leq 1}$ in μ obtained by projecting such an integral curve $\gamma_{F_\Delta}(t)_{0 \leq t \leq 1}$, for some

generating set Δ of W , will be called a *critical F-curve*.) In this case, we choose E so that there exists an immersion of $[0,1] \times [0,1]$ into W with the following properties: 1) the image of $[0,1] \times [0,1]$ under i is equal to $\pi_W(E)$, 2) $\pi_W(E \cap \partial D) = i([0,1] \times \{0\})$ and 3) for every $t_0 \in [0,1]$, $i(\{1/2\} \times \{t_0\}) = \gamma(t_0)$ and $\gamma(t)_{0 \leq t \leq 1} \cap i([0,1] \times \{t_0\})$ is equal to either $\gamma(t_0)$ or $i([0,1] \times \{t_0\})$. We also ensure that $\mu \cap \pi_W(E)$ is the union of $\gamma(t)_{0 \leq t \leq 1}$ with finitely many pairwise disjoint compact arcs, each containing precisely one triple point which is also contained in γ . If γ has length N , we say the extension of D to contain E is a *critical F-extension of length N* . In this case, the corresponding splitting is called a *critical F-splitting of length N* . (Note that, by definition, each critical F-splitting of W is along a critical F-curve γ ; that is, it corresponds to an F-extension along a curve $\gamma_{F,\Delta}$, as described above, for some generating set Δ .)

We can also modify an element D of Δ by replacing it with a proper subset of itself. If this subset is connected and has finitely many boundary components, and if the new Δ also satisfies condition ii) for a generating set, then we refer to this modification of D as a *contraction*. Note that the connectedness condition ensures that a contraction does not change the cardinality of the generating set. This is also true for F-extensions provided that the elements of Δ are contained in distinct leaves of F . In such cases, each F-extension can be reversed by a contraction.

If a contraction of some $D \in \Delta$ yields another generating set for (F, ϕ) , then it corresponds to a *pinching* of W . Specifically, such a contraction deletes some open subset S of D . If B is the component of $M-W$ corresponding to D , then there exist two subsets S_+ and S_- of ∂B corresponding to S which can be identified to partially collapse B . In other words, we can pinch these pieces of W together to obtain the branched surface that is generating by Δ after the contraction.

Any branched surface obtained from a pinching of W also carries F , so each pinching is the inverse of an F-splitting. A *critical pinching of W of length N* is the inverse

of a critical F-splitting of length N ; i.e., a critical pinching of length N yields a critical F-curve in the branch set of the resulting branched surface that has length N .)

Throughout, we consider finitely many successive modifications of a branched surface W that result in a string W_0, \dots, W_n of branched surfaces where $W_0 = W$ and for some natural number N and every $i \leq n$, the branched surface W_{i+1} is obtained from W_i by either a critical F-splitting of length at most N (i.e. by a splitting along a critical F-curve in W_i of length at most N) or a critical pinching of length at most N . In such cases, we say the branched surface W_n is obtained by *modifying W by n successive critical F-splittings and pinchings, each of length at most N* . We let $[W]_{F,N}$ represent the set of branched surfaces that can be obtained from W by at most N successive critical F-splittings and pinchings, each of length at most N . Since, for every natural number N , the number of critical F-curves in the branch set of W having length N is finite, up to parameterization, the set $[W]_{F,N}$ is finite. (Here, we do not distinguish between two critical F-curves γ' and γ'' if they agree except possibly along open neighborhoods of their respective initial points or terminal points that do not meet μ , since in this case F-splittings along γ and γ' respectively yield diffeomorphic branched surfaces.)

In addition to F-extensions and contractions, we can also change a generating set Δ for (F, ϕ) by replacing some element D with another compact integral surface D' of F such that D flows continuously, along orbits of ϕ , onto D' . As noted earlier, this does not necessarily change the branched surface W generated by Δ . A substitution in Δ that does not change W will be called an *F-bumping*, since it usually involves moving a generating surface to a nearby leaf. (Note that a bumping could change the foliation of $N(W)$ induced by F .)

III An equivalence on foliations transverse to the same flow

In this section, we use our modification techniques from the previous section to define an equivalence relation on foliations of any closed 3-manifold M that are transverse to a fixed nonsingular C^1 flow ϕ .

An appropriate relation should ensure that representatives of the same equivalence class bear some similarity to each other. This is the case for foliations that shadow each other in some branched surface. However, the shadowing property is often stronger than we need. So for each natural number N , we introduce a weaker notion, called *N-equivalence*. For N sufficiently large, foliations that are N -equivalent will share some important topological properties. In fact, we use this to define a pseudometric on foliations transverse to ϕ under which nearby foliations are topologically similar. This then allows us to define an equivalence relation on foliations transverse to ϕ which subsumes the shadowing property.

Degree N equivalence on foliations transverse to the same flow. Given a foliation F transverse to ϕ , we say a generating set for (F, ϕ) is *standard minimal for (F, ϕ)* if it consists of embedded disks and if no other standard branched surface can be constructed from F and ϕ using a generating set consisting of fewer disks (although it is possible that some branched surface could be constructed from F and ϕ using a generating set consisting of fewer embedded surfaces, some of which are not simply connected). It is worth noting that all elements in a standard minimal generating set for (F, ϕ) are contained in distinct leaves of F (since, otherwise, we could extend some element of Δ in its leaf so that it merges with another to form one large generating disk.)

A branched surface is *standard minimal for (F, ϕ)* if it has a generating set that is standard minimal for (F, ϕ) . There is at least one such branched surface for any pair (F, ϕ) , since we can always find a generating set consisting of embedded disks (see Section I).

Let $\Omega(F,\phi)=\{W: W \text{ is a branched surface with a connected branch set that is standard minimal for } (F,\phi) \text{ such that no other branched surface of this type has fewer triple points than does } W\}$. This set is nonempty for every pair (F,ϕ) since any standard minimal generating set Δ for (F,ϕ) can be modified by F -extensions so that its branch set is connected without creating a nontrivial loop in Δ or increasing the cardinality of Δ .

Now for each natural number N , we let $\Omega_N(F,\phi)$ be the set of all branched surfaces that can be obtained from a branched surface in $\Omega(F,\phi)$ by at most N successive critical F -splittings, each of length at most N . (Since F -extensions can merge pieces of the same generating disk, the branched surfaces in $\Omega_N(F,\phi)$ might not all be standard.) By definition, $\Omega_N(F,\phi)$ is contained in $\bigcup_{W \in \Omega(F,\phi)} [W]_{F,N}$. Furthermore, $\Omega(F,\phi)$ is contained in $\Omega_N(F,\phi)$ and $\Omega_N(F,\phi)$ is contained in $\Omega_{N+1}(F,\phi)$, for every natural number N . In the proof of Proposition 3.1, we shall show that the cardinality of $\Omega_N(F,\phi)$ is finite for every N .

Definition

Let ϕ be a nonsingular flow on M . Given foliations F and G transverse to ϕ and a natural number N , we say F and G are N -equivalent, and write $F \sim_N G$, if $\Omega_N(F,\phi)=\Omega_N(G,\phi)$. Clearly this relation is reflexive, symmetric and transitive.

Since for every N , the set $\Omega_N(F,\phi)$ (and hence the N -equivalence class of a foliation F) depends on the transverse flow ϕ , we shall henceforth fix a nonsingular flow ϕ on M . For example, $F \sim_N G$ shall mean that both F and G are transverse to this ϕ and that $\Omega_N(F,\phi)=\Omega_N(G,\phi)$.

Using the definitions of $\Omega_N(F,\phi)$ and $\Omega_N(G,\phi)$, it is straightforward to verify that $F \sim_N G$ if and only if G is carried by every $W \in \Omega_N(F,\phi)$ and F is carried by every $V \in \Omega_N(G,\phi)$.

Proposition 3.1

Let ϕ be any nonsingular flow on M . For every natural number N , there are at most countably many N -equivalence classes for foliations transverse to ϕ and each can be associated with a distinct finite collection of simplicial complexes.

Proof: Given a standard branched surface W with a connected branch set μ , the intersection $W\epsilon$ of W with a small regular neighborhood of μ in the ambient manifold M is obtained by piecing together local neighborhoods of the triple points, each of which is modeled on either Figure 1.3 or Figure 1.4. (We glue these local models together along the Y-shaped components of their boundaries in a manner dictated by the branch set.) The branched surface W can then be constructed by gluing the boundaries of planar surfaces homeomorphic to the sectors of W to $\partial W\epsilon$. Since $\partial W = \emptyset$ and W has finitely many triple points, (i.e., the set μ is a finite connected graph), it follows that there are countably many possibilities for such a W , up to diffeomorphism. In fact, for each nonnegative integer k , the collection Σ_k of all standard branched surfaces with a connected branch set and exactly k triple points has finite cardinality. So its power set $\mathcal{P}(\Sigma_k)$ also has finite cardinality. Now, for every pair (F, ϕ) , the set $\Omega(F, \phi)$ is an element of $\mathcal{P}(\Sigma_k)$ for some k . In particular, the cardinality of $\Omega(F, \phi)$ is finite. Furthermore, the number of possibilities for the finite set $\Omega(F, \phi)$, over all pairs (F, ϕ) , is countable. For any branched surface W and every natural number N , there are only finitely branched surfaces that can be obtained by splitting W along a curve in its branch set whose length is at most N . So the set $\Omega_N(F, \phi)$ is finite, for every pair (F, ϕ) and every natural number N , and the number of possibilities for the set $\Omega_N(F, \phi)$ over all pairs (F, ϕ) is countable. The result now follows from the definition of N -equivalence for foliations transverse to ϕ . \square

The following proposition will be useful in Section IV where we investigate topological properties that are shared by foliations in the same equivalence class, for N sufficiently large.

Proposition 3.2.

Given a foliation F and a transverse flow ϕ , if for some natural number N , there exists a branched surface $W \in \Omega_N(F, \phi)$ with a topological property P , then any foliation G that is N -equivalent to F has property P . If, in addition, W is standard, then property P is C^1 -stable for G .

(Here we are using the C^1 metric on foliations defined by Hirsch [Hi] in which a nearby foliation is obtained by perturbing the tangent bundle to the leaves to another integrable plane field.)

Proof:

Suppose $W \in \Omega_N(F, \phi)$ has property P for some natural number N ; that is, every foliation carried by W has property P . If G is a foliation that is N -equivalent to F , then $W \in \Omega_N(G, \phi)$. It follows that G is carried by W , hence has property P . If W is standard, then all foliations sufficiently close to G are also carried by W [Sh1]; that is, each foliation within some $\varepsilon > 0$ of G , in the C^1 metric, is carried by W . (There may also be foliations carried by W' that are not within ε of G .) So property P is C^1 -stable for G . \square

Using the density of Smale flows in the C^0 topology of nonsingular flows [OI], we shall assume that the flow ϕ is Smale for the rest of this section. (Recall that a nonsingular flow ϕ on a manifold is called a *Smale flow* provided 1) the chain recurrent set \mathcal{R} of ϕ has hyperbolic structure and topological dimension one, and 2) for any two points x and y in \mathcal{R} , the stable manifold of x and the unstable manifold of y intersect transversely. For a general discussion of Smale flows, see [Fr1]. Sullivan [Su] also gives a nice visual description of the dynamics of these flows.) However, the only property of Smale flows that we shall use is the following: there exists a closed invariant one-dimensional subset \mathcal{R} of M such that each orbit of ϕ contains in its limit set some orbit in \mathcal{R} . (When ϕ is Smale, we can choose \mathcal{R} to be the chain recurrent set.) We show that for flows with this property, a foliation G is carried by every element of $\Omega_N(F, \phi)$ if for some $W \in \Omega(F, \phi)$ and S sufficiently large, G is carried by every element of $[W]_{F, S}$. So to verify N -equivalence of two foliations, it is

enough to show that for some branched surface $W \in \Omega(F, \phi) \cap \Omega(G, \phi)$ and some S sufficiently large, $[W]_{F,S} = [W]_{G,S}$; in other words, it is often sufficient to focus on a single branched surface and consider only finitely many splittings and pinchings of that W . First, we shall need the following:

Theorem 3.3

Let F be a foliation of M and ϕ be a Smale flow transverse to F . Any generating set for a branched surface $W \in \Omega(F, \phi)$ can be modified to obtain a generating set for any other $V \in \Omega(F, \phi)$ by successive F -extensions, contractions and bumpings.

Proof: Let $\Delta = \{D_i\}_{1 \leq i \leq n}$ and $X = \{C_i\}_{1 \leq i \leq n}$ be standard generating sets for branched surfaces $W \in \Omega(F, \phi)$ and $V \in \Omega(F, \phi)$ respectively. Without loss of generality, we can assume that the chain recurrent set \mathcal{R} for ϕ does not meet $\partial\Delta$. Specifically, since \mathcal{R} has topological dimension one, we can take an arbitrarily small extension of any generating disk $D_i \in \Delta$ within its leaf to obtain a disk D_i^* whose boundary misses \mathcal{R} ; \mathcal{R} being closed implies that there exists a open collar neighborhood of ∂D_i^* missing \mathcal{R} ; hence, general position arguments allow us to perturb D_i^* so that the conditions for a generating set are still satisfied after our extension.

Bumping elements of Δ to nearby leaves if necessary, we can also assume that $D_i \cap C_j = \emptyset$ for all $i, j \leq n$. So when we cut the manifold M open along the elements of X to obtain $N_X(V)$, each element of Δ becomes embedded in the interior of $N_X(V)$, transverse to the fibers. (Each element C_i of X yields a boundary component B_i of $N_X(V)$.) Given $i \leq n$, we could eliminate all branchings of V along the positive side of $\pi_V(D_i)$ an F -splitting. More precisely, choose $k \leq n$ such that D_i flows continuously forward, along fibers of $N_X(V)$, onto another integral surface of F_X that intersects B_k , before possibly meeting $\partial N_X(V) - B_k$; then use an F -extension of C_k to include this surface. After the extension, B_k meets the upper boundary of each fiber through D_i ; in particular, D_i flows continuously, along fibers of $N_X(V)$, onto an integral surface contained in B_k . Suppose that C_k can be extended further so that this is also the case for some $D_j, j \neq i$. In this case, the new C_k can be used to replace

both D_i and D_j in Δ . Moreover, if the extended C_k is not simply connected, we can first extend it further so that its boundary misses \mathcal{R} , and then contract to a disk by deleting a finite collection of compact strips in C_k that miss \mathcal{R} . Since each orbit of ϕ limits on an orbit in \mathcal{R} , condition ii) for a generating set will still be satisfied by $(\Delta - \{D_i, D_j\}) \cup \{C_k\}$ after we delete these strips. But then, after some slight additional modification of C_k , $(\Delta - \{D_i, D_j\}) \cup \{C_k\}$ is a standard generating set for (F, ϕ) , contradicting our assumption that Δ is standard minimal for (F, ϕ) . So for each $j \neq i$, C_k cannot be F -extended so that the corresponding component of $\partial N_X(V)$ meets the upper boundary of every fiber of $N_X(V)$ through D_j . Reindexing the elements of the original X if necessary, we can therefore assume that for every $i \leq n$, D_i flows continuously, along fibers of $N_X(V)$, onto an integral surface intersecting B_i before possibly meeting $\partial N_X(V) - B_i$. That is, D_i flows continuously forward, along orbits of ϕ , onto an integral surface D_i' intersecting C_i and no orbit segment from D_i to D_i' meets $X - \{C_i\}$.

We claim that D_i can be modified by contractions and F -extensions so that it flows continuously and injectively onto a disk contained in the same leaf of F as C_i .

Proof of claim: Our concern will be those parts of D_i that flow into D_i before flowing into D_i' .

Let Φ be the collection of fibers of $N_X(V)$ through D_i . (In particular, every element of Φ is an orbit segment of ϕ .) General position arguments allow us to modify D_i by F -extensions so that there are only finitely many fibers in the set Φ containing more than one point in ∂D_i . (Since there exists a neighborhood of ∂D_i in its leaf that misses \mathcal{R} , we can ensure that ∂D_i still misses \mathcal{R} after these modifications.) This ensures that the image ∂' of ∂D_i in D_i' is a closed connected one-manifold except at finitely many points where it self intersects. (It also ensures that D_i' has finitely many boundary components.)

The set ∂' partitions D_i' into finitely many regions whose interiors, which we shall refer to as *sections of D_i'* , are pairwise disjoint. The preimage in D_i of any section Γ' of D_i' (i.e. the set of points that map onto Γ' when we flow D_i continuously onto D_i') has finitely

many components, each of which flows continuously and injectively onto Γ' . For example, suppose D_i is as shown in Figure 3.1 and that the transverse flow is perpendicular to the page. A possibility for D_i' is shown in Figure 3.2. It is divided into 17 sections by the image ∂' of ∂D_i . One of these sections is shaded; its preimage in D_i has three components.

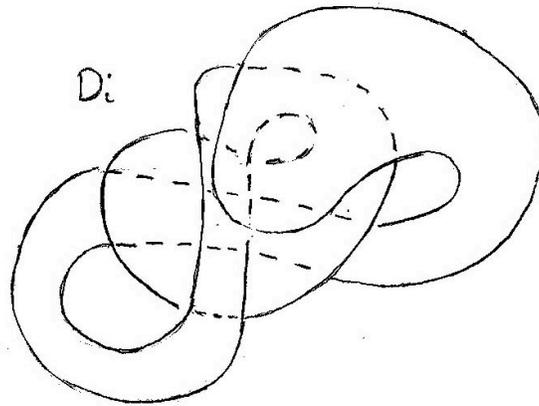


Figure 3.1

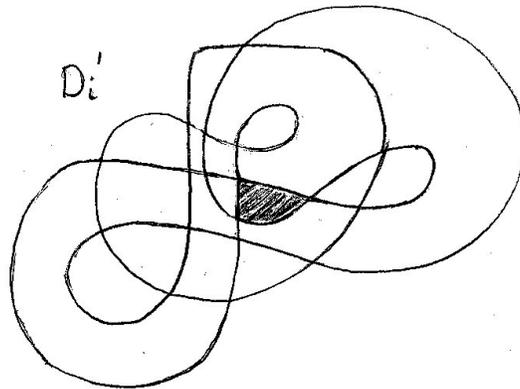


Figure 3.2

Let $\Gamma_0' \dots \Gamma_p'$ be the sections of D_i' and for every $k \leq p$, let Γ_k be a component of the preimage of Γ_k' in D_i . Our first objective will be to flow the components of $\text{cl}(\Gamma_0) \cup \dots \cup \text{cl}(\Gamma_k)$ (either forward or backward), along fibers in the set Φ , to obtain a connected integral surface of F that intersects D_i . We then use extensions and contractions

get from D_i to this surface, and argue that repeating this procedure finitely many times gives a D_i that flows injectively onto its image D_i' .

We can choose our indices so that $\text{cl}(\Gamma_0') \cup \text{cl}(\Gamma_1')$ is connected. In this case, either $\text{cl}\Gamma_0 \cap \text{cl}\Gamma_1$ is connected or $\partial\Gamma_1$ contains a point x that flows (either forward or backward) along some fiber in the set Φ onto some point y in $\partial\Gamma_0'$. In the latter case, there exists an integral surface Γ_1^* such that $\text{cl}(\Gamma_1)$ flows continuously onto $\text{cl}(\Gamma_1^*)$ and $\text{cl}(\Gamma_0) \cup \text{cl}(\Gamma_1^*)$ is connected. Specifically, D_i can be mapped continuously (either forward or backward) along fibers in the set Φ , onto another integral surface of F in such a way that x maps to y , so this is also the case for any portion of D_i , say $\text{cl}(\Gamma_1)$. (It is worth noting that $\text{cl}(\Gamma_1)$ does not necessarily flow *injectively* onto $\text{cl}(\Gamma_1^*)$.) So in either case, D_i intersects a connected integral surface S_1 (equal to $\text{cl}(\Gamma_0) \cup \text{cl}(\Gamma_1)$ or $\text{cl}(\Gamma_0) \cup \text{cl}(\Gamma_1^*)$ respectively) with the property that for every $z \in \text{cl}(\Gamma_0) \cup \text{cl}(\Gamma_1)$, the fiber of Φ through z meets S_1 .

Proceeding inductively, assume that for some $k \leq p$, $\text{cl}(\Gamma_0' \cup \dots \cup \Gamma_{k-1}')$ is connected and that there exists a closed and connected integral surface S_{k-1} of F containing $\text{cl}(\Gamma_0)$ with the property that for every $j \leq k-1$, either $\text{cl}(\Gamma_j)$ is contained in S_{k-1} or $\text{cl}(\Gamma_j)$ flows continuously (either backward or forward), along fibers in the set Φ , into S_{k-1} . Reindexing the set $\{\Gamma_k, \dots, \Gamma_p\}$ if necessary, we can assume that $\text{cl}(\Gamma_0' \cup \dots \cup \Gamma_k')$ is connected. So either $S_{k-1} \cup \text{cl}(\Gamma_k)$ is connected (in which case we let S_k be this surface) or some fiber in the set Φ meets both ∂S_{k-1} and $\partial\Gamma_k$. In the latter case, we can argue, as above, that there exists an integral surface Γ_k^* such that $\text{cl}(\Gamma_k)$ flows continuously (either backward or forward) along fibers in the set Φ onto $\text{cl}(\Gamma_k^*)$ and $S_{k-1} \cup \text{cl}(\Gamma_k^*)$ is connected. We then let $S_k = S_{k-1} \cup \text{cl}(\Gamma_k^*)$. So, in either case, there exists a closed integral surface S_k containing Γ_0 so that for every $j \leq k$, either $\text{cl}(\Gamma_j)$ is contained in S_k or $\text{cl}(\Gamma_j)$ flows continuously, along fibers in the set Φ , into S_k . By induction, these conditions are satisfied when $k=p$. Since Γ_0 is contained in both S_p and D_i , we can extend D_i so that it contains the surface S_p and then contract it to delete all points not contained in S_p .

Henceforth, to reflect the fact that D_i modifies by F-extensions and contractions to give the surface S_p , which we shall use \tilde{D}_i , rather than S_p , to denote this surface. (For example, Figure 3.3 shows a possibility for \tilde{D}_i when D_i is as in Figure 3.1.) By the way we constructed \tilde{D}_i ($=S_p$), condition ii for a generating set is still satisfied after these modifications. If \tilde{D}_i is not simply connected, then we remove a finite collection K of compact arcs to ensure that it is. We can choose each arc in K to be the continuous image of some arc in ∂D_i (as we flow it partially forward or backward along fibers in the set Φ). This ensures that $K \cap \mathcal{R} = \emptyset$, which means that after we remove K from \tilde{D}_i , we still have condition ii for a generating set satisfied; specifically, each orbit of ϕ limits on an orbit of \mathcal{R} meeting $\text{int}\Delta - (K \cap \text{int}\Delta)$. Note that \tilde{D}_i might not be a generating disk after these contractions (in fact, it might not even be closed). But since its boundary misses \mathcal{R} , it could be contracted further so that all conditions for a generating set are again satisfied. However, for simplicity in our argument, we do not do this until the last stage of our modification process.

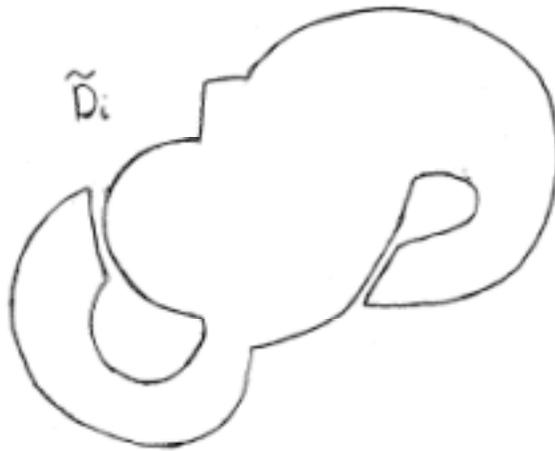


Figure 3.3

Now, \tilde{D}_i flows continuously, along fibers in the set Φ , onto a surface \tilde{D}_i' of F contained in the same leaf of F as C_i . Just as we observed in the initial case, $\partial \tilde{D}_i$ flows continuously (along orbits of Φ) onto a connected subset of \tilde{D}_i' which partitions it into finitely many regions whose interiors are disjoint. So we can define *sections* for \tilde{D}_i' .

If we consider the set of points in D_i' (\tilde{D}_i') that are contained in an element of Φ through ∂D_i ($\partial \tilde{D}_i'$ respectively), we see this set divides D_i' (\tilde{D}_i') into finitely many regions whose interiors are disjoint, and the closure of each section of D_i' (\tilde{D}_i' respectively) is the union of such regions. Furthermore, our construction of \tilde{D}_i' ensures that it has fewer regions of this type than does D_i' . So, continuing in this manner, we would eventually get some \tilde{D}_i' with only one such region, hence one only section.

The claim is proved.

So after F-extensions and contractions of D_i , we can flow it continuously and injectively, along orbits of ϕ , onto a disk D_i' in the same leaf as C_i , for each $i \leq n$. If some orbit from D_i into D_i' meets $\Delta - \{D_i\}$, then we could flow D_i continuously forward so that it intersects another element of Δ (since D_i is a disk). Since $\partial \Delta \cap \mathcal{R} = \emptyset$, we could then modify the union of D_i with this element by contractions, as described earlier, so that it is a single generating disk, contracting our assumption that Δ is standard minimal. It follows that a bumping of D_i takes it onto D_i' .

We can then use F-extensions to get C_i contained in D_i' , for every $i \leq n$. Subsequent contractions in $\{D_i'\}_{1 \leq i \leq n}$ yields X . \square

Corollary 3.5

Let F be a foliation of M and ϕ be a Smale flow transverse to F . There exists a natural number S_F , such that any foliation that is carried by each element of $[W]_{F, S_F + N}$, for some $W \in \Omega(F, \phi)$ and some natural number N , is also carried by each element of $\Omega_N(F, \phi)$.

Proof: Let W and V be two elements of the finite set $\Omega(F, \phi)$ with generating sets Δ and X respectively. We first find a number $S_{(W, V)}$ such that every foliation carried by $[W]_{F, S_{(W, V)}}$ is also carried by V . The idea is as follows. By Theorem 3.4, we can use F-extensions and contractions of Δ to obtain a generating set that bumps onto a set Δ' , which can be similarly modified to get X . We will consider only those critical F-curves that are

involved in one of these F -extensions or that arise as a result of one of these contractions. In particular, we let $S_{(W,V)}$ be the sum of the cardinality of this set of critical F -curves with the length of its longest element.

To begin, note that when modifying Δ to obtain Δ' , we can, in fact, do all F -extensions first. In particular, the extensions yield finitely many disjoint integral surfaces, each of which contains a disk that bumps onto an element of Δ' . So let \mathcal{C} be the finite set of integral surfaces of F corresponding to the F -extensions used to get from Δ to Δ' . In other words, \mathcal{C} is the finite set of integral surfaces of F that we adjoin to the elements of Δ in order to get a generating set that modifies by contractions and bumpings to give Δ' . For each $E \in \mathcal{C}$, let Σ_E be the set of critical F -curves in W that are projections of curves in E containing at most one loop. (This set is finite up to parameterization.) In particular, if a foliation G is carried by each branched surface obtained after an F -splitting of W along some curve in Σ_E , then it is carried by the branched surface obtained after extending the element of Δ intersecting E to include E .

Now, given two critical F -curves γ_1 and γ_2 in Σ_E corresponding to integral curves $(\gamma_1)_{F\Delta}$ and $(\gamma_2)_{F\Delta}$ (respectively) of F_Δ contained in E , the branched surface W_1 obtained after extending an element of Δ to contain $(\gamma_1)_{F\Delta}$ contains finitely many pairwise disjoint critical F -curves corresponding γ_2 . (Specifically, each critical F -curve of W_1 corresponding to γ_2 is a parameterization of some component of $\pi_{W_1}\{(\gamma_2)_{F\Delta} - [E_1 \cap (\gamma_2)_{F\Delta}]\}$ where E_1 is an open collar neighborhood of $(\gamma_1)_{F\Delta}$ in its leaf, as described in Section II.) So we can find a finite string of branched surfaces $W = W_0, \dots, W_K$ such that for every $0 \leq i < K$, W_{i+1} is obtained from W_i by splitting along some critical F -curve in W_i , and any foliation carried by W_K is also carried by the branched surface obtained when we extend the element of Δ intersecting E to include E . We let $S_E = K + \max\{\text{length of splitting used to get from } W_i \text{ to } W_{i+1}; 0 \leq i < K\}$. Any foliation carried by each element of $[W]_{F,SE}$ is carried by the branched surface obtained after extending the element of Δ intersecting E to include E . Now let $S_1 = \max\{S_E; E \in \mathcal{C}\}$,

Let W' be the branched surface generated by Δ' . As above, we find a finite set \mathcal{C}' of integral surfaces of F that we adjoin to elements of Δ' to obtain a generating set that contracts to X . For each $E' \in \mathcal{C}'$ we can define $\Sigma_{E'}$ as above and find a number $S_{E'}$ with the property that any foliation that is carried by $[W']_{F, \Sigma_{E'}}$ is carried by the branched surface obtained after extending the element of Δ' intersecting E' to include E' . Now let $S_2 = \max\{S_{E'} : E' \in \mathcal{C}'\}$.

Now, the generating set Δ' is obtained by first extending Δ along surfaces in Σ and then taking finitely many contractions (followed by bumpings). So we also consider those critical contractions necessary to create each of the critical F -curves in $\bigcup_{E' \in \mathcal{C}'} \Sigma_{E'}$. Let T_1 be the number of these contractions and let T_2 be an upper bound on their length.

For the pair (W, V) we let $S_{(W, V)} = S_1 + S_2 + T_1 + T_2$. It follows that any foliation carried by $[W]_{F, S_{(W, V)}}$ is carried by V .

Then, setting $S_F = \{\max S_{(W, V)} : W \in \Omega(F, \phi), V \in \Omega(F, \phi)\}$ we get the result. \square

So, to show N -equivalence of two foliations F and G transverse to ϕ for some natural number N , it is often enough to focus on some $W \in \Omega(F, \phi) \cap \Omega(G, \phi)$ and consider only finitely many splittings and pinchings of this W . In particular, it suffices to show that $[W]_{F, S} = [W]_{G, S}$ for $S = \max\{S_F, S_G\} + N$.

The equivalence relation on foliations transverse to ϕ . We can use N -equivalence to define a pseudometric on the set of foliations transverse to ϕ , where

$\|F - G\| = 1/\sup\{N : F \sim_N G\}$ if $\sup\{N : F \sim_N G\}$ exists, and $\|F - G\| = 0$ otherwise. We then define two foliations F and G transverse to some flow ϕ to be *b-equivalent*, and write $F \sim G$, if $\|F - G\| = 0$. (So our pseudometric induces a metric on the set of b -equivalence classes.) Since $F \sim_{N+1} G$ implies that $F \sim_N G$, foliations F and G are b -equivalent if and only if $F \sim_N G$ for every N . As with N -equivalence, b -equivalence depends on the flow ϕ . So when we

write $F \sim G$, we shall be assuming a fixed flow ϕ and that both F and G are transverse to ϕ . (In Section V, we define a stronger equivalence relation on foliations that does not depend on a particular transverse flow.)

It is worth noting that there are often infinitely many foliations in the same b-equivalence class. As a simple example, consider the collection $\{F_k ; k \geq 2\}$ of foliations of the torus such that each element F_k has 2 Reeb components, k compact leaves and is carried by the branched 1-manifold W in Figure 2.2. For every N and every k , $\{W\} = \Omega_N(F_k, \phi)$, so all of these foliations are b-equivalent.

If $F \sim G$, then F and G shadow each other in every $W \in \Omega(F, \phi)$. This will be exploited in Section IV to show that b-equivalent foliations share many important topological properties. However, the shadowing property can be difficult to verify for certain foliations and it is usually stronger than we need. In fact, we will see that when F has a topological property P , we can often produce a branched surface with property P by splitting some $W \in \Omega(F, \phi)$ along finitely many critical F -curves. In this case, there exists a natural number N such that all foliations that are N -equivalent to F have property P .

IV. Topological properties shared by equivalent foliations

In this section, we discuss topological properties shared by foliations that shadow each other in some branched surface. In particular, these properties are shared by all b-equivalent foliations and, in most cases, by foliations that are sufficiently close, up to b-equivalence. As before, ϕ will be a nonsingular flow on a closed orientable 3-manifold M and F will be a foliation of M and that is everywhere transverse to ϕ . All equivalence relations will be with respect to ϕ .

Any branched surface W carrying a foliation F can be modified by finitely many critical F -splittings to obtain a branched surface W' with the property that no foliation carried by W' has more dead-end components than does F . In particular, it was shown in [Sh3] that there is a finite set Σ of compact surfaces (not necessarily connected) embedded

in $N(W)$ such that for any foliation carried by W , the boundary of each dead-end component is isotopic, in $N(W)$, to an element of Σ . Furthermore, if Δ is a generating set for W , any element of Σ that does not bound a dead-end component of F_Δ can be deleted by an F -extension; in other words, we can use finitely many critical F -splittings of W to obtain a branched surface W' such that all elements of the corresponding set Σ' bound a dead-end component of F_Δ . This gives the following:

Proposition 4.1

Given a foliation F , there exists a natural number N such that no foliation that is N -equivalent to F has more dead-end components than does F ; in particular, if F is taut and $G \sim_N F$, then G is taut.

Now suppose that F has a compact leaf C . We can construct $W \in \Omega(F, \phi)$ so that C embeds in the interior of $N(W)$. If G is b -equivalent to F , then F shadows G in W . So by [Sh4] we have:

Proposition 4.2

If a foliation F has a compact leaf C and $G \sim F$, then G has a compact leaf that is isotopic to C .

Next, we consider the case where F is *R-covered*: that is, F lifts to a foliation of the universal cover with leaf space homeomorphic to the real line \mathbb{R} . (Recall that the *leaf space* of a foliation is the quotient space obtained by identifying points in the ambient manifold that lie on the same leaf.) Foliations with the \mathbb{R} -covered property are particularly nice in the sense that they are completely determined by the induced action of $\pi_1(M)$ on \mathbb{R} [So]. (For more on the \mathbb{R} -covered property, see [Ba 1-2], [Ca], [Fe 1-4], [Gh], [Pl 1-2]). If F is \mathbb{R} -covered and $W \in \Omega(F, \phi)$, we can find a subset Γ_W of smooth curves in W with the following property: For any non \mathbb{R} -covered foliation carried by W there exists a curve $\gamma(t)_{0 \leq t \leq 1} \in \Gamma_W$ that lifts to a curve $\hat{\gamma}(t)_{0 \leq t \leq 1}$ in \hat{W} whose ends branch into the same side of the projections of two nonseparable leaves \hat{A} and \hat{B} respectively (i.e. \hat{A} and \hat{B} correspond to distinct

nonseparable points in the leaf space of the universal cover and $\hat{\gamma}(0) \in \pi_w^{-1}(\hat{A})$, $\hat{\gamma}(1) \in \pi_w^{-1}(\hat{B})$ [Go-Sh]. See Figure 4.1.

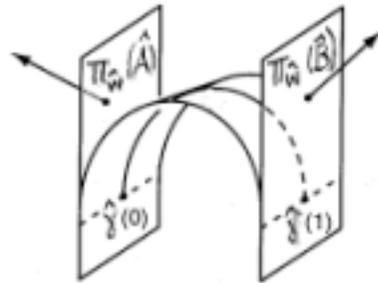


Figure 4.1

By [Sh2] we have the following:

Proposition 4.3

If a foliation F is R -covered and $G \sim F$, then G is R -covered. Furthermore, if for some $W \in \Omega(F, \phi)$ the set Γ_W can be chosen to have finite cardinality, then there exists an N such that every foliation that is N -equivalent to F is R -covered; in this case, the R -covered property is stable for these foliations.

We now turn our attention to geometric entropy and the growth of leaves in F . Before stating the proposition, recall that a *weight system* on a branched surface W is an assignment of a nonnegative real number or *weight* to each sector so that the weights satisfy the obvious additive condition with respect to the branch set. See Figure 4.2. For example, each branched surface admits a *trivial weight system* where each weight is 0. (For details, see [Fl-Oe].)

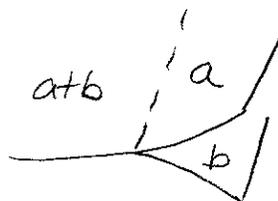


Figure 4.2

Next, we show the following:

Theorem 4.4

If F is carried by a branched surface W with only the trivial weight system, then every leaf of F has exponential growth and F has positive entropy. Furthermore, if $W \in \Omega_N(F, \phi)$ for some N , then these properties hold for all foliations that are N -equivalent to F ; if W is standard, then these properties are stable for all foliations that are N -equivalent to F .

Proof:

If F contains a leaf L with nonexponential growth, then there exists a nontrivial holonomy invariant measure on (M, F) which is finite on compact sets (and has support contained in $\text{cl}(L)$) ([Pl 3]). Likewise, if F has zero entropy, then F has a nontrivial holonomy invariant measure which is finite on compact sets ([Gh-La-Wa]). In either case, the holonomy invariant measure induces a nontrivial weight system on any branched surface W carrying F when we fix a generating set Δ and let the weight of a sector be the measure of any fiber of $N_\Delta(W)$ over the interior of that sector. So, if a branched surface W admits only the trivial weight system, then all foliations carried by it have exponential growth and positive entropy. The result now follows from Proposition 3.2. \square

Conversely, if $W \in \Omega_N(F, \phi)$ has a nontrivial weight system, we can use an algorithm in [Sh3] to find a lower bound k on the depth of any foliation carried by W . Specifically, for any foliation G , if G is N -equivalent to F and all leaves of G are at finite depth, then G contains a leaf at depth k .

Finally, it seems likely that the topological properties considered in this section are not the only ones that are shared by foliations that are N -equivalent, when N is sufficiently large. Such foliations yield foliations of some $N(W)$ whose leaves come close to shadowing each other; so the topological dynamics of these foliations are similar over long time intervals.

V. An equivalence relation independent of ϕ .

Although the relations \sim_N and \sim are defined only for foliations transverse to the same flow ϕ , we can define more general relations that partition all foliations of a 3-manifold M into equivalence classes as follows.

Given a foliation F , let Σ_F a subset of nonsingular flows on M that are transverse to F such that for every $\phi \in \Sigma_F$, each standard minimal branched surface W carrying (F, ϕ) has the fewest number of generating disks possible. That is, no branched surface in $\Omega(F, \phi')$ for some ϕ' transverse to F can be constructed using fewer generating disks than are necessary to generate an element of $\Omega(F, \phi)$. We then restrict to a subset Σ_F^* of Σ_F so that for any $\psi \in \Sigma_F$ and any $\phi \in \Sigma_F^*$, the branched surfaces in $\Omega(F, \psi)$ do not have fewer triple points than do the branched surfaces in $\Omega(F, \phi)$. It follows that the set $\Omega^*(F) = \{W : W \in \Omega(F, \phi) \text{ for some } \phi \in \Sigma_F^*\}$ has finite cardinality. Now let $\Omega_N^*(F)$ be the set of all branched surfaces that can be obtained from a branched surface $W \in \Omega^*(F)$ by at most successive N critical F -splittings, each of length at most N . Define relations on the set of foliations of M as follows:

Definition

Given foliations F and G and a natural number N , we say F and G are N^* -equivalent, and write $F \sim_{N^*} G$, if $\Omega_N^*(F) = \Omega_N^*(G)$.

This can be used to define a pseudometric on all foliations of M , as in Section III. We then define b^* -equivalence for foliations of M in a manner analogous to b -equivalence above, but using this new pseudometric.

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